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Steady filtration problems with seawater intrusion: Macro-hybrid penalized finite element approximations

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SUMMARY

Macro-hybrid penalized finite element approximations are studied for steady filtration problems with seawater intrusion. On the basis of nonoverlapping domain decompositions with vertical interfaces, sections of coastal aquifers are decomposed into subsystems with simpler geometries and small scales, interconnected via transmission conditions of pressure and flux continuity. Corresponding local penalized formulations are derived from the global penalized variational formulation of the two-free boundary flow problem, with continuity transmission conditions modelled variationally in a dual sense. Then, macro-hybrid finite element approximations are derived for the system, defined on independent subdomain grids. Parallel relaxation penalty-duality algorithms are proposed from fixed-point problem characterizations. Numerical experiments exemplify the macro-hybrid penalized theory, showing a good agreement with previous primal conforming penalized finite element approximations (*Comput. Methods Appl. Mech. Engng.* 2000; **190**:609–624). Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: open flow in porous media; seawater intrusion problem; free boundary problem; domain decomposition method; macro-hybrid finite element; proximal-point algorithm

1. INTRODUCTION

The variational study and numerical simulation of free boundary problems associated to filtration phenomena through open aquifers was initiated by Baiocchi *et al.* [1, 2]. The strategy there, valid only for very restrictive geometries, was to transform the original problem into an obstacle one, to which approximation techniques for monotone variational and quasivariational inequalities are applicable [3, 4]. For problems with general geometry, where Baiocchi's transform does not apply, an especial non-subdifferential variational formulation was introduced by Brezis–Kinderlehrer–Stampacchia [5] and Alt [6, 7], and corresponding numerical studies were performed in References [8–10] to problems with one free boundary.

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In the case of open coastal aquifers of a general geometry, where two free boundaries appear as interfaces of saturation and seawater intrusion, the variational approach of Brezis–Kinderlehrer–Stampacchia was applied in Reference [11], and analysed in Reference [12], extending some of the qualitative results of Carrillo-Menendez and Chipot [13, 14]. Then, from the penalized model introduced in Reference [12] for analysis purposes, we have started the study of computational models, considering first in Reference [15] primal conforming penalized finite element schemes for simulating steady open flows with seawater intrusion.

The purpose of this paper is to continue the study of finite element schemes for filtration processes in coastal aquifers with general sections. Our approach here will be the application of nonoverlapping domain decomposition techniques to the system, in order to localize subsystems with simpler geometry and small scale. This will permit us, apart from treating large-scale systems, to introduce local conforming penalized finite element approximations for each subsystem, defined on independent subdomain grids. Then, in this manner, globally nonconforming macro-hybrid penalized finite element approximations are produced susceptible of being resolved in parallel. Due to the applied fundamental variational approach [12], based on the strong maximum principle for elliptic problems, the domain decomposition of the system is restricted to have only vertical interfaces; otherwise, local variational subproblems cannot be realized.

Outstanding aspects of this work are the non-subdifferential variational formulation of the problem, and the dual proximation treatment of the transmission conditions of pressure and flux continuity across the vertical interfaces, that communicate hydraulically the several local subsystems produced by nonoverlapping domain decompositions. Corresponding local variational formulations are derived from the global penalized variational formulation of the two-free boundary flow problem, analysed in Reference [12], with transmission conditions modelled variationally in a dual sense. Then, following References [8–10, 15], and according to the theory of resolvent methods for macro-hybrid variational inequalities [16–18], parallel relaxation penalty-duality algorithms are constructed from proximation fixed-point problem characterizations. Specifically, the interface iterative algorithms are given via augmented Uzawa type approximations, which produce Robin type interface transmission [19] as a fictitious overlapping [20]. Further, since local models are of a nonlinear advection dominated type, for uniform convergence the Tabata upwind approximation [21] is applied, as in Reference [9] for filtration without seawater intrusion.

At the end of the work, some numerical experiments exemplify the macro-hybrid theory, showing a good agreement with previous primal conforming penalized finite element approximations studied in Reference [15]. Also, asymptotic pressure and interface flow error behaviour are confirmed as well as convergent augmented synchronizing processes through two- and three-field relaxation proximal-point algorithms. Furthermore, numerical optimal exact penalization parameters are experimentally determined.

2. THE GLOBAL PHYSICAL AND VARIATIONAL MODELS

2.1. The physical model

Let us consider a bounded connected section of an open coastal aquifer with general geometry, denoted by $\Omega \subset \Re^2$, with a Lipschitz continuous boundary $\partial\Omega$, in which the pressure distribution, p, is to be determined for steady flow processes under the presence of fresh water reservoirs, the intrusion of seawater, and the gravity effect (see Figure 1). Assuming homogeneity and isotropy for the porous media, and the constitutivity of a Darcian incompressible fluid flow with velocity field $\mathbf{u} = -\nabla(p+y)$, where parameters are normalized, the principle of mass conservation, pressure and flux boundary conditions, as well as interface free boundary conditions lead to the following two-free boundary problem.

Find the pressure field p and the fresh water wet set (flow domain) $A \subset \Omega$ such that the pressure equation is satisfied,

$$-\Delta(p+y) = 0 \quad \text{in } A \tag{1}$$

subject to the pressure and impervious boundary conditions

$$\begin{aligned} p + y &= \hat{h} && \text{on } \Gamma_{\mathrm{fw}} = \partial\Omega_{\mathrm{fw}} \\ p + y &= \rho y && \text{on } \Gamma_{\mathrm{sw}} = \partial A \cap \partial\Omega_{\mathrm{sw}} \\ -\nabla(p+y)\cdot\mathbf{n} &= 0 && \text{on } \Gamma_{\mathrm{i}} = \partial A \cap \partial\Omega_{\mathrm{i}} \end{aligned} \tag{2}$$

the seepage pressure condition and flux constraint

$$\left.\begin{aligned} p + y &= y \\ -\nabla(p+y)\cdot\mathbf{n} &\geqslant 0 \end{aligned}\right\} \quad \text{on } \Gamma_{\mathrm{a}} = \partial A \cap \partial\Omega_{\mathrm{a}} \tag{3}$$

and the interface free boundary pressure-flux conditions

$$\begin{aligned} &\left.\begin{aligned} p + y &= y \\ -\nabla(p+y)\cdot\mathbf{n} &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_{\mathrm{sfb}} \subset \Omega \\ &\left.\begin{aligned} p + y &= \rho y \\ -\nabla(p+y)\cdot\mathbf{n} &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_{\mathrm{ifb}} \subset \Omega \end{aligned} \tag{4}$$

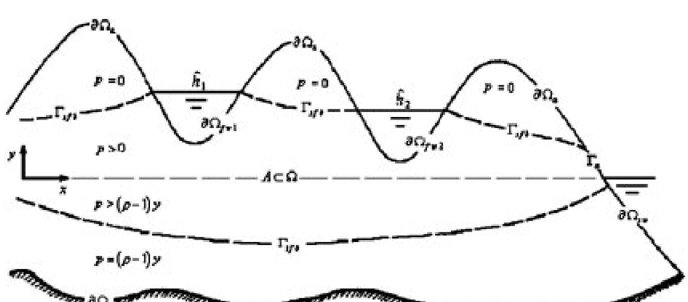

Figure 1. A section of an open coastal aquifer.

Here, $\hat{h} > 0$ denotes generically the fresh water level of the reservoirs, and the negative parameter

$$\rho \equiv \frac{\rho_{\mathrm{f}} - \rho_{\mathrm{s}}}{\rho_{\mathrm{f}}} < 0 \tag{5}$$

is defined in terms of the fresh water and seawater mass densities, ρ_{f} and ρ_{s}. Further, the boundary of the flow domain A, with outwards unit normal $\mathbf{n}$, $\partial A = \Gamma_{\mathrm{fw}} \cup \Gamma_{\mathrm{sw}} \cup \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{a}} \cup \Gamma_{\mathrm{sfb}} \cup \Gamma_{\mathrm{ifb}}$, is defined relative to the boundary of the aquifer section Ω, $\partial\Omega = \Omega_{\mathrm{fw}} \cup \Omega_{\mathrm{sw}} \cup \Omega_{\mathrm{i}} \cup \Omega_{\mathrm{a}}$, with subscripts standing for: 'fw' = part under fresh water, 'sw' = part under seawater, 'i' = impervious part, 'a' = part in contact with the open air, 'sfb' = saturation interface free boundary, and 'ifb' = fresh–seawater interface free boundary.

For the variational formulation of this global problem, according to Reference [12], the following geometrical hypotheses are required:

$$\text{(H1)} \quad \hat{b} \geqslant \hat{h}/\rho$$

where $\hat{b}$ denotes, generically, the lowest ordinate of the bottom of the reservoirs, and

$$\text{(H2)} \quad n_y \equiv \nabla y \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{i}}$$

the former guaranteeing that the fresh water reservoirs cannot be reached by the seawater intrusion, and the latter imposing the (possibly empty) impervious boundary $\Gamma_{\mathrm{i}} = \partial A \cap \partial\Omega_{\mathrm{i}}$ to be vertical. Then, the functions p and $p - (\rho - 1)y$, harmonic in A, satisfy the maximum principle: they have positive Dirichlet traces and inflow Neumann values at the boundary ∂A. Consequently, the pressure field satisfies the inequality

$$p > \varphi \equiv \max\ (0, (\rho - 1)y) = \begin{cases} 0, & y \geqslant 0 \\ (\rho - 1)y, & y < 0 \end{cases} \quad \text{in } A \tag{6}$$

and the section of the aquifer is characterized by

$$\Omega = A^{+} \cup A \cup A^{-} \tag{7}$$

where

$$\begin{aligned} A^{+} &= [p = 0] \equiv \{(x, y) \in \Omega : p(x, y) = 0\} \\ A &= [p > \varphi] \equiv \{(x, y) \in \Omega : p(x, y) > \varphi\} \\ A^{-} &= [p = (\rho - 1)y] \equiv \{(x, y) \in \Omega : p(x, y) = (\rho - 1)y\} \end{aligned} \tag{8}$$

Therefore, any solution (p, A) of the global problem (1)–(7) is such that

$$p + \{(1 - \rho)H_0(p) + \rho H_0(p - \varphi)\}y = \begin{cases} 0 & \text{in } A^{+} \\ p + y & \text{in } A \\ 0 & \text{in } A^{-} \end{cases} \tag{9}$$

where H_0 is the Heaviside function: $H_0(\theta) = 0$, $\theta \leqslant 0$, and $H_0(\theta) = 1$, $\theta > 0$, and φ is the continuous extension of the obstacle function (6) to Ω by zero in A^{+} and by $(\rho - 1)y$ in A^{-}.

2.2. The variational model

From the above results, the physical problem can be extended variationally from A to all of Ω in the following sense, with an inequality structure due to the presence of the velocity seepage constraint $(3)_2$ [12].

$$\text{(V)}\quad \begin{cases} \text{Find } (p,\lambda(p))\in S\times L^{\infty}(\Omega) \\ \displaystyle\int_{\Omega}[\nabla p+\{(1-\rho)\lambda(p)+\rho\lambda(p-\varphi)\}\nabla y]\cdot\nabla v\,\mathrm{d}\Omega\leqslant 0 \quad \forall v\in K \end{cases}$$

where λ is such that,

$$\lambda(\theta)\in H(\theta)=\begin{cases}\{1\}, & \theta>0\\ [0,1], & \theta=0\\ \{0\}, & \theta<0\end{cases} \tag{10}$$

and the admissibility subsets are defined by

$$\begin{aligned} S&=\{q\in H^1(\Omega): q\geqslant\varphi \text{ a.e. in } \Omega,\ \gamma q=g \text{ a.e. on } \partial\Omega_{\mathrm{fw}}\cup\partial\Omega_{\mathrm{sw}}\cup\partial\Omega_{\mathrm{a}}\}\\ K&=\{v\in H^1(\Omega):\gamma v=0 \text{ a.e. on } \partial\Omega_{\mathrm{fw}}\cup\partial\Omega_{\mathrm{sw}},\ \gamma v\geqslant 0 \text{ a.e. on } \partial\Omega_{\mathrm{a}}\}\end{aligned} \tag{11}$$

with Dirichlet trace operator denoted by γ and boundary function

$$g=\begin{cases}\hat{h}-y & \text{on } \partial\Omega_{\mathrm{fw}}\\ (\rho-1)y & \text{on } \partial\Omega_{\mathrm{sw}}\\ 0 & \text{on } \partial\Omega_{\mathrm{a}}\end{cases} \tag{12}$$

Global variational problem (V) has a solution, and this is unique whenever every molecule of the fluid inside the aquifer is connected to the reservoirs (see Reference [12] for details and other qualitative properties).

3. PENALIZED VARIATIONAL FORMULATIONS

We next introduce the penalized formulation of variational problem (V), studied in Reference [12], and, on the basis of nonoverlapping domain decomposition techniques, we derive the corresponding macro-hybrid variational formulation.

3.1. Penalized problem

Following Reference [12], we consider the penalization of global problem (V),

$$(\mathrm{V}_{\varepsilon})\quad \begin{cases} \text{Find } p_{\varepsilon}\in S_{\varepsilon} \\ \displaystyle\int_{\Omega}[\nabla p_{\varepsilon}+\mu_{\varphi_{\varepsilon}}(p_{\varepsilon})\nabla y]\cdot\nabla v\,\mathrm{d}\Omega=0 \quad \forall v\in K_0 \end{cases}$$

where the function $\mu_{\varphi_\varepsilon}$ is defined by

$$\mu_{\varphi_\varepsilon}(p_\varepsilon) = (1-\rho)H_\varepsilon(p_\varepsilon) + \rho H_\varepsilon(p_\varepsilon - \varphi_\varepsilon) \tag{13}$$

and the solution and variation subsets are

$$\begin{aligned} S_\varepsilon &= \{q \in H^1(\Omega) : \gamma q = g_\varepsilon \text{ a.e. on } \partial\Omega_\mathrm{D}\} \\ K_0 &= \{v \in H^1(\Omega) : \gamma v = 0 \text{ a.e. on } \partial\Omega_\mathrm{D}\} \end{aligned} \tag{14}$$

with Dirichlet boundary $\partial\Omega_\mathrm{D} = \partial\Omega_\mathrm{fw} \cup \partial\Omega_\mathrm{sw} \cup \partial\Omega_\mathrm{a}$. Here, $\varepsilon > 0$ denoting the penalization parameter, multivalued Heaviside function (10), extended obstacle function (6), and Dirichlet boundary data (12) are, respectively, penalized by

$$H_\varepsilon(\theta) = \begin{cases} 1, & \theta > \varepsilon \\ \theta/\varepsilon, & 0 \leqslant \theta \leqslant \varepsilon \\ 0, & \theta < 0 \end{cases} \tag{15}$$

$$\varphi_\varepsilon = \begin{cases} \varepsilon \exp((\rho-1)y/\varepsilon), & y \geqslant 0 \\ (\rho-1)y + \varepsilon, & y < 0 \end{cases} \quad \text{in } \Omega \tag{16}$$

and

$$g_\varepsilon = g + \gamma(\varphi_\varepsilon - \varphi) \quad \text{on } \partial\Omega_\mathrm{D} \tag{17}$$

Penalized problem (V_ε) possesses a unique solution, $p_\varepsilon \in S_\varepsilon$, such that

$$p_\varepsilon \geqslant \varphi_\varepsilon \quad \text{in } \Omega \tag{18}$$

and, as $\varepsilon \to 0$, it is convergent weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to a solution of the original global problem (V) (see Reference [12]).

3.2. Macro-hybrid formulation

We now proceed to macro-hybridize the penalized variational problem (V_ε) on the basis of nonoverlapping domain decompositions, localizing in this manner the two-free boundary problem into simpler subdomains (see Figure 2). Let the spatial domain Ω be decomposed in terms of connected disjoint subdomains $\{\Omega_e\}$,

$$\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e \tag{19}$$

with internal boundaries and interfaces denoted by

$$\begin{aligned} \Gamma_e &= \partial\Omega_e \cap \Omega, \quad e = 1, \dots, E \\ \Gamma_{ef} &= \Gamma_e \cap \Gamma_f, \quad 1 \leqslant e < f \leqslant E \end{aligned} \tag{20}$$

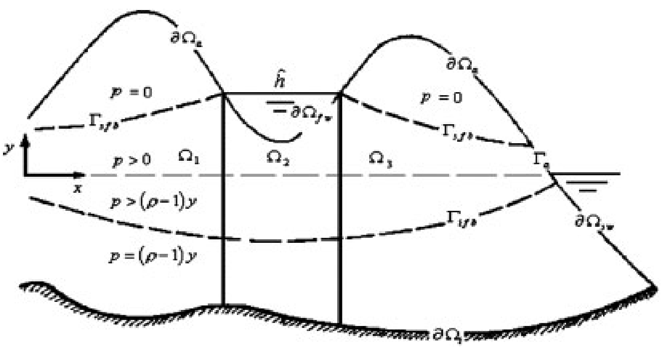

Figure 2. A decomposed section of an open coastal aquifer with vertical interfaces.

and assumed to be Lipschitz continuous. Then, for nonoverlapping domain decompositions (19) and (20), the macro-hybrid formulation of problem (V_ε) is obtained according to the following results:

- Let $[\gamma_{\Gamma_e}]$ denote the Dirichlet internal boundary trace operators that satisfy the fundamental macro-hybrid compatibility condition

$$\text{(C1)} \quad \gamma_{\Gamma_e} \in \mathscr{L}(H^1(\Omega_e), H^{1/2}(\Gamma_e)), \quad e=1,2,\ldots,E, \text{ are surjective}$$

  From the sum of the local integration by parts formulae,

$$\int_{\Omega_e} \partial_i v_e \phi \,\mathrm{d}\Omega = -\int_{\Omega_e} v_e \partial_i \phi \,\mathrm{d}\Omega + \int_{\partial\Omega_e} \gamma_{\Gamma_e} v_e \gamma_{\Gamma_e} \phi n_{e,i} \,\mathrm{d}\partial\Omega$$
$$v_e \in H^1(\Omega_e), \phi \in C_0^\infty(\Omega), \quad e=1,2,\ldots,E$$

  ∂_i denoting Cartesian ith-partial differentiation, and according to the concept of generalized differentiation, it readily follows that the global solution space $H^1(\Omega)$ has the decomposed characterization (cf. Reference [22] at a conforming finite element level)

$$H^1(\Omega) \simeq \left\{ \{v_e\} \in \prod_{e=1}^{E} H^1(\Omega_e) : \{\gamma_{\Gamma_e} v_e\} \in \mathbf{Q}_\mathrm{D} \right\} \tag{21}$$

  where $\mathbf{Q}_\mathrm{D}$ is the primal admissibility subspace of interface continuity,

$$\mathbf{Q}_\mathrm{D} = \left\{ \{\xi_e\} \in \prod_{e=1}^{E} H^{1/2}(\Gamma_e) : \xi_e = \xi_f \text{ a.e. on } \Gamma_{ef}, \ 1 \leqslant e < f \leqslant E \right\} \tag{22}$$

  Hence, the internal boundary traces of the solution restrictions of problem (V_ε), $\{\gamma_{\Gamma_e} p_{\varepsilon|\Omega_e}\}$, belong to $\mathbf{Q}_\mathrm{D}$.
- On the other hand, localizing the penalized problem (V_ε) in terms of variations from $C_0^\infty(\Omega_e)$, it follows that the restrictions of its solutions, $\{p_\varepsilon^e = p_{\varepsilon|\Omega_e}\}$, satisfy the local

equations

$$-\text{div}(\nabla p_\varepsilon^e + \mu_{\varphi_\varepsilon}(p_\varepsilon^e)\nabla y) = 0, \text{ a.e. in } \Omega_e, \quad e = 1, \ldots, E \tag{23}$$

- Furthermore, formulating Equations (23) variationally with respect to subdomain restrictions of variations from $H_0^1(\Omega)$, summing up over the index e, and taking into account compatibility condition (C1), it turns out that the local restrictions of penalized solutions satisfy in addition the internal boundary flux admissibility condition,

$$\sum_{e=1}^{E} \int_{\Gamma_e} -[\nabla p_\varepsilon^e + \mu_{\varphi_\varepsilon}(p_\varepsilon^e)\nabla y] \cdot \mathbf{n}_e \, \xi_e \, \mathrm{d}\Gamma = 0 \quad \forall \{\xi_e\} \in \mathbf{Q}_\mathrm{D} \tag{24}$$

  Here, $\int_{\Gamma_e} \cdot \, \mathrm{d}\Gamma$ denotes the duality pairing for the internal boundary space $H^{1/2}(\Gamma_e)$ and its dual $H^{-1/2}(\Gamma_e)$. Notice that in fact, under L^2-regularity, this condition implies the penalized flux continuity transmission condition of the filtration problem: for $1 \leqslant e < f \leqslant E$,

$$-[\nabla p_\varepsilon^e + \mu_{\varphi_\varepsilon}(p_\varepsilon^e)\nabla y] \cdot \mathbf{n}_e = [\nabla p_\varepsilon^f + \mu_{\varphi_\varepsilon}(p_\varepsilon^f)\nabla y] \cdot \mathbf{n}_f, \quad \text{a.e. on } \Gamma_{ef} \tag{25}$$

- That is, the internal boundary fluxes,

$$w_\varepsilon^e = -[\nabla p_\varepsilon^e + \mu_{\varphi_\varepsilon}(p_\varepsilon^e)\nabla y] \cdot \mathbf{n}_e \in H^{-1/2}(\Gamma_e), \quad e = 1, \ldots, E \tag{26}$$

  are orthogonal to the primal transmission subspace $\mathbf{Q}_\mathrm{D}$ in the dual sense: they belong to the dual admissibility subspace of 'interface flux continuity'

$$\mathbf{Q}_\mathrm{N}^* = \left\{ \{\chi_e\} \in \prod_{e=1}^{E} H^{-1/2}(\Gamma_e) : \sum_{e=1}^{E} \int_{\Gamma_e} \chi_e \xi_e \, \mathrm{d}\Gamma = 0 \ \forall \{\xi_e\} \in \mathbf{Q}_\mathrm{D} \right\} \tag{27}$$

  Note that the transmission admissibility subspaces $\mathbf{Q}_\mathrm{D}$ and $\mathbf{Q}_\mathrm{N}^*$ are orthogonal to each other in the dual sense.

From the above decomposition results, relative to (19) and (20), we can conclude that subdomain restrictions of penalized solutions of problem (V_ε), $\{p_\varepsilon^e = p_{\varepsilon|\Omega_e}\}$, in conjunction with their internal boundary fluxes (26), are necessarily solutions of the macro-hybrid penalized problem,

$$(\mathrm{MH}_\varepsilon) \quad \begin{cases} \text{Find } \ p_\varepsilon^e \in S_\varepsilon^e \quad \text{for } e = 1, \ldots, E \\ \displaystyle\int_{\Omega_e} [\nabla p_\varepsilon^e + \mu_{\varphi_\varepsilon}(p_\varepsilon^e)\nabla y] \cdot \nabla v \, \mathrm{d}\Omega = -\int_{\Gamma_e} w_\varepsilon^e \gamma_{\Gamma_e} v \, \mathrm{d}\Gamma \quad \forall v \in K_0^e \\ \text{and } \{w_\varepsilon^e\} \in \mathbf{Q}_\mathrm{N}^* \text{ satisfying the synchronizing condition} \\ \displaystyle\sum_{e=1}^{E} \int_{\Gamma_e} (\chi_e - w_\varepsilon^e)\gamma_{\Gamma_e} p_\varepsilon^e \, \mathrm{d}\Gamma = 0 \quad \forall \{\chi_e\} \in \mathbf{Q}_\mathrm{N}^* \end{cases}$$

where S_ε^e and K_0^e, $e = 1, 2, \ldots, E$, are the local versions of subsets (14)

$$\begin{aligned} S_\varepsilon^e &= \{q \in H^1(\Omega_e) : \gamma q = g_\varepsilon \text{ a.e. on } \partial\Omega_e \cap \partial\Omega_\mathrm{D}\} \\ K_0^e &= \{v \in H^1(\Omega_e) : \gamma v = 0 \text{ a.e. on } \partial\Omega_e \cap \partial\Omega_\mathrm{D}\} \end{aligned} \tag{28}$$

Conversely, summing up local variational equations of macro-hybrid problem (MH_ε), with variations from $K_0 \subset \prod_{e=1}^{E} K_0^e$, and taking into account that the synchronizing condition implies $\gamma_{\Gamma_e} p_\varepsilon^e \in \mathbf{Q}_\mathrm{D}$, we can also conclude that macro-hybrid solutions $\{p_\varepsilon^e\} \in \prod_{e=1}^{E} S_\varepsilon^e$ conform to solutions of the original penalized filtration problem (V_ε) since compatibility condition (C1) holds. Therefore, variational penalized problems (V_ε) and (MH_ε) are equivalent to each other in a solvability sense, and, as a corollary, macro-hybrid penalized problem (MH_ε) has a unique solution. Note that the uniqueness of macro-hybrid solutions follows from the uniqueness of penalized primal solutions and the injectivity of the transpose trace operators $\gamma_{\Gamma_e}^{\mathrm{T}} \in \mathscr{L}(H^{-1/2}(\Gamma_e),(H^1(\Omega_e))')$, $e=1,2,\ldots,E$, implied by condition (C1).

We observe that due to the linear structure of subspace $\mathbf{Q}_{\mathrm{N}}^*$, the synchronizing condition of problem (MH_ε) is equivalent to the classical hybrid linear variational equation [23–25]

$$\{w_\varepsilon^e\} \in \mathbf{Q}_{\mathrm{N}}^* : \sum_{e=1}^{E} \int_{\Gamma_e} \chi_e \gamma_{\Gamma_e} p_\varepsilon^e \, \mathrm{d}\Gamma = 0 \quad \forall \{\chi_e\} \in \mathbf{Q}_{\mathrm{N}}^* \tag{29}$$

Furthermore, this synchronizing condition is in fact equivalent to the dual subdifferential equation, that in turn is equivalent to its primal subdifferential equation,

$$\{\gamma_{\Gamma_e} p_\varepsilon^e\} \in \boldsymbol{\partial} I_{Q_{\mathrm{N}}^*}(\{w_\varepsilon^e\}) \iff \{w_\varepsilon^e\} \in \boldsymbol{\partial} I_{Q_{\mathrm{D}}}(\{\gamma_{\Gamma_e} p_\varepsilon^e\}) \tag{30}$$

where $I_{Q_{\mathrm{N}}^*} : \prod_{e=1}^{E} H^{-1/2}(\Gamma_e) \to \Re \cup \{+\infty\}$ is the indicator functional of dual subspace $\mathbf{Q}_{\mathrm{N}}^*$ (zero in $\mathbf{Q}_{\mathrm{N}}^*$, otherwise $+\infty$), with conjugate $I_{Q_{\mathrm{D}}} : \prod_{e=1}^{E} H^{1/2}(\Gamma_e) \to \Re \cup \{+\infty\}$, the indicator functional of primal subspace $\mathbf{Q}_{\mathrm{D}}$ (zero in $\mathbf{Q}_{\mathrm{D}}$, otherwise $+\infty$) (cf., e.g. References [26, 27]); i.e. subspaces $\mathbf{Q}_{\mathrm{D}}$ and $\mathbf{Q}_{\mathrm{N}}^*$ are orthogonal to each other in the dual sense (they are the polar subspace of each other). Interpretation (30) will play a fundamental role in the proximation characterization of synchronization, when deriving proximal-point algorithms for finite element discretizations in Section 5.

4. MACRO-HYBRID FINITE ELEMENT APPROXIMATIONS

We now pass to introduce spatial discretizations of macro-hybrid penalized problem (MH_ε) in terms of local stabilized conforming finite element approximations, which can be naturally defined on independent subdomain and interface grids. In this manner, we shall produce nonconforming multidomain or macro-hybrid penalized finite element approximations of the original two-free boundary filtration problem (V).

For $e=1,2,\ldots,E$, let

$$V_{h_e} = \mathrm{span}\{\phi_1^e, \phi_2^e, \ldots, \phi_{m_{h_e}}^e\} \subset H^1(\Omega_e) \tag{31}$$

be local pressure finite element spaces of dimension m_{h_e}, defined on regular triangulations of the nonoverlapping subdomains of decomposition (19) and (20), and in general with not matching traces at the interfaces. Further, let

$$B_{\tilde{h}_e} = \mathrm{span}\{\psi_1^e, \psi_2^e, \ldots, \psi_{n_{\tilde{h}_e}}^e\} \subset L^2(\Gamma_e) \subset H^{-1/2}(\Gamma_e) \tag{32}$$

be relatively local internal flux finite element spaces of dimension $n_{\tilde{h}_e}$, defined on independent internal boundary grids. We shall assume that $\{V_{h_e}\}_{h_e>0}$ and $\{B_{\tilde{h}_e}\}_{\tilde{h}_e>0}$ are families of conforming finite element approximations of local spaces $H^1(\Omega_e)$ and $L^2(\Gamma_e)$, for $e=1,2,\ldots,E$, respectively [22]. Then, expressing approximate local penalized pressure fields by

$$p_{\varepsilon,h_e} = \sum_{j=1}^{m_{h_e}} \alpha^e_{\varepsilon,j} \phi^e_j \in V_{h_e} \tag{33}$$

and approximate internal boundary fluxes by

$$w_{\varepsilon,\tilde{h}_e} = \sum_{k=1}^{n_{\tilde{h}_e}} \delta^e_{\varepsilon,k} \psi^e_k \in B_{\tilde{h}_e} \tag{34}$$

we associate to macro-hybrid penalized problem $(\mathrm{MH}_\varepsilon)$ the following finite element discretization.

$$(\mathrm{MH}_{\varepsilon,h,\tilde{h}}) \quad \begin{cases} \text{Find } \boldsymbol{\alpha}^e_\varepsilon \in \mathbf{S}^{h_e}_\varepsilon \quad \text{for } e = 1, \ldots, E \\ \{\mathbf{A}^e \boldsymbol{\alpha}^e_\varepsilon + (1-\rho)\mathbf{H}^e_\varepsilon(\boldsymbol{\alpha}^e_\varepsilon) + \rho \mathbf{H}^e_\varepsilon(\boldsymbol{\alpha}^e_\varepsilon - \boldsymbol{\eta}^e_\varepsilon)\} \cdot \boldsymbol{\beta} = -\Lambda^{e\mathrm{T}} \boldsymbol{\delta}^e_\varepsilon \cdot \boldsymbol{\beta} \quad \forall \boldsymbol{\beta} \in \mathbf{K}^{h_e}_0 \\ \text{and } \{\boldsymbol{\delta}^e_\varepsilon\} \in \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}} \text{ satisfying the synchronizing condition} \\ \sum_{e=1}^{E} (\mathbf{v}_e - \boldsymbol{\delta}^e_\varepsilon) \cdot \Lambda^e \boldsymbol{\alpha}^e_\varepsilon = 0 \quad \forall \{\mathbf{v}_e\} \in \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}} \end{cases}$$

Here, for $e = 1, 2, \ldots, E$, $\mathbf{S}^{h_e}_\varepsilon$ and $\mathbf{K}^{h_e}_0$ stand for $\Re^{m_{h_e}}$ versions of given discrete admissibility sets $S_{\varepsilon,h_e} \subset V_{h_e}$ and $K_{0,h_e} \subset V_{h_e}$, and $\boldsymbol{\eta}^e_\varepsilon \in \Re^{m_{h_e}}$ corresponds to the coordinate vector of an approximate local penalized obstacle function $\varphi_{\varepsilon,h_e} \in V_{h_e}$. In particular, appropriate definitions can be given as usual in terms of corresponding interpolants of the Dirichlet boundary prescribed function and the local obstacle functions, as we will do in Section 6 (cf. Reference [15]). Further, local matrix $\mathbf{A}^e$, nonlinear discrete vector function $\mathbf{H}^e_\varepsilon$ and coupling matrix Λ^e are defined by, for $1 \leqslant i, j \leqslant m_{h_e}$, $1 \leqslant k \leqslant n_{\tilde{h}_e}$,

$$\begin{aligned} A^e_{ij} &= \int_{\Omega_e} \nabla \phi^e_j \cdot \nabla \phi^e_i \, \mathrm{d}\Omega \\ H^e_{\varepsilon,i}(\boldsymbol{\beta}) &= \int_{\Omega_e} H_\varepsilon \left(\sum_{k=1}^{m_{h_e}} \beta_k \phi^e_k \right) \nabla y \cdot \nabla \phi^e_i \, \mathrm{d}\Omega, \ \ \boldsymbol{\beta} \in \Re^{m_{h_e}} \\ \Lambda^e_{ki} &= \int_{\Gamma_e} \psi^e_k \gamma_{\Gamma_e} \phi^e_i \, \mathrm{d}\Gamma \end{aligned} \tag{35}$$

Regarding the discrete version of the dual transmission admissibility subspace $\mathbf{Q}^*_\mathrm{N}$ of (27), we shall assume that hybrid approximations (32) are coincident at the interfaces in the following sense. Expressing the internal boundaries by $\Gamma_e = \bigcup_{cd \in \mathscr{I}_e} \Gamma_{cd}$, $e = 1, \ldots, E$, with $\mathscr{I}_e$ the corresponding e-interface index set, we assume that the internal boundary finite element spaces are characterized by $B_{\tilde{h}_e} = \bigoplus_{cd \in \mathscr{I}_e} B_{\tilde{h}_{cd}}$, with respect to conforming interface spaces $B_{\tilde{h}_{cd}} = [\psi^{cd}_1, \psi^{cd}_2, \ldots, \psi^{cd}_{k_{\tilde{h}_{cd}}}] \subset L^2(\Gamma_{cd}) \subset H^{-1/2}(\Gamma_{cd})$, such that $n_{\tilde{h}_e} = \sum_{cd \in \mathscr{I}_e} k_{\tilde{h}_{cd}}$. In this manner, by virtue of the L^2-regularity of the discrete internal boundary fluxes, the $\Re^{n_{\tilde{h}_e}}$ version of the discrete dual transmission admissibility subspace turns out to be (cf. (25))

$$\mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}} = \left\{ \{\mathbf{v}_e\} \in \prod_{e=1}^{E} \Re^{n_{\tilde{h}_e}} : v^e_{cd,k} = -v^f_{cd,k},\ k=1,\ldots,k_{\tilde{h}_{cd}}, cd \in \mathscr{I}_e \cap \mathscr{I}_f,\ 1 \leqslant e < f \leqslant E \right\} \tag{36}$$

and then, consequently, the $\Re^{n_{\tilde{h}_e}}$ version of the discrete primal transmission admissibility subspace $\mathbf{Q}_{\mathrm{D}}$ of (22), as orthogonal to $\mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$ is given by

$$\mathbf{Q}_{\mathrm{D}_{\tilde{h}}} = \left\{ \{\boldsymbol{\mu}_e\} \in \prod_{e=1}^{E} \Re^{n_{\tilde{h}_e}} : \mu^e_{cd,k} = \mu^f_{cd,k},\ k=1,\ldots,k_{\tilde{h}_{cd}}, cd \in \mathscr{I}_e \cap \mathscr{I}_f,\ 1 \leqslant e < f \leqslant E \right\} \tag{37}$$

4.1. *Well-posedness of problem* ($\mathrm{MH}_{\varepsilon,h,\tilde{h}}$)

For the well-posedness of the macro-hybrid discrete problem ($\mathrm{MH}_{\varepsilon,h,\tilde{h}}$), we shall assume that the locally conforming finite element approximations (31) and (32) satisfy the corresponding discrete version of the macro-hybrid compatibility condition (C1),

$$(\mathrm{C1}_{h,\tilde{h}}) \quad \Lambda^e \in \mathscr{L}(\Re^{m_{h_e}}, \Re^{n_{\tilde{h}_e}}), \quad e=1,2,\ldots,E, \text{ are surjective}$$

which is equivalent to the lower boundedness of the transpose matrices $\Lambda^{e\mathrm{T}} \in \mathscr{L}(\Re^{n_{\tilde{h}_e}}, \Re^{m_{h_e}})$, $e=1,2,\ldots,E$, that in turn corresponds to the classical inf–sup condition [23–25]: there exist constants $c_e > 0$, $e=1,2,\ldots,E$, such that

$$\inf_{v_e \in \Re^{n_{\tilde{h}_e}} \setminus \{0\}} \ \sup_{\beta_e \in \Re^{m_{h_e}} \setminus \{0\}} \frac{\Lambda^{e\mathrm{T}} \mathbf{v}_e \cdot \boldsymbol{\beta}_e}{\|\mathbf{v}_e\|_{\Re^{n_{\tilde{h}_e}}} \|\boldsymbol{\beta}_e\|_{\Re^{m_{h_e}}}} \geqslant c_e \tag{38}$$

Then, under compatibility condition ($\mathrm{C1}_{h,\tilde{h}}$), we can conclude, as in the case of the continuous penalized problems (MH_ε) and (V_ε), the solvability of macro-hybrid discrete problem ($\mathrm{MH}_{\varepsilon,h,\tilde{h}}$) through the solvability of the corresponding primal discrete problem ($\mathrm{V}_{\varepsilon,h,\tilde{h}}$). Indeed, the subdifferential form of the discrete synchronizing condition of problem ($\mathrm{MH}_{\varepsilon,h,\tilde{h}}$) satisfies the equivalent subdifferential relations

$$\begin{aligned}
\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\} \in \boldsymbol{\partial} I_{Q^*_{\mathrm{N}_{\tilde{h}}}}(\{\boldsymbol{\delta}^e_\varepsilon\}) &\iff \{\boldsymbol{\delta}^e_\varepsilon\} \in \boldsymbol{\partial} I_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\}) \\
&\iff \{\Lambda^{e\mathrm{T}} \boldsymbol{\delta}^e_\varepsilon\} \in \boldsymbol{\partial}(I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e])(\{\boldsymbol{\alpha}^e_\varepsilon\})
\end{aligned} \tag{39}$$

The first equivalence holds as in (30) by duality. The second one is a consequence of the compatibility condition ($\mathrm{C1}_{h,\tilde{h}}$) since under $[\Lambda^e]$-surjectivity their primal variational inequalities are equivalent to each other:

$$\begin{aligned}
&\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}} : I_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\boldsymbol{\mu}_e\}) \geqslant I_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\}) + \{\boldsymbol{\delta}^e_\varepsilon\} \cdot (\{\boldsymbol{\mu}_e\} - \{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\}) \quad \forall \{\boldsymbol{\mu}_e\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}} \\
&\{\boldsymbol{\alpha}^e_\varepsilon\} \in \mathscr{D}(I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e]) : I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e](\{\boldsymbol{\beta}_e\}) \\
&\quad \geqslant I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e](\{\boldsymbol{\alpha}^e_\varepsilon\}) + \{\Lambda^{e\mathrm{T}} \boldsymbol{\delta}^e_\varepsilon\} \cdot (\{\boldsymbol{\beta}_e\} - \{\boldsymbol{\alpha}^e_\varepsilon\}) \quad \forall \{\boldsymbol{\beta}_e\} \in \mathscr{D}(I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e])
\end{aligned} \tag{40}$$

Therefore, the local primal components of solutions of macro-hybrid discrete problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$ conform to solutions of the global primal discrete problem

$$(\mathrm{V}_{\varepsilon,h,\tilde{h}}) \quad \begin{cases} \text{Find } \{\boldsymbol{\alpha}_\varepsilon^e\} \in \prod_{e=1}^{E} \mathbf{S}_\varepsilon^{h_e} \text{ with } \{\Lambda^e \boldsymbol{\alpha}_\varepsilon^e\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}} \\ \sum_{e=1}^{E} (\mathbf{A}^e \boldsymbol{\alpha}_\varepsilon^e + (1-\rho)\mathbf{H}_\varepsilon^e(\boldsymbol{\alpha}_\varepsilon^e) + \rho \mathbf{H}_\varepsilon^e(\boldsymbol{\alpha}_\varepsilon^e - \boldsymbol{\eta}_\varepsilon^e)) \cdot \boldsymbol{\beta}_e = 0 \\ \forall \{\boldsymbol{\beta}_e\} \in \prod_{e=1}^{E} \mathbf{K}_0^{h_e} \text{ with } \{\Lambda^e \boldsymbol{\beta}_e\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}} \end{cases}$$

which defines macro-hybrid finite element discrete solutions of continuous problem (V_ε), in general nonconforming since for non-matching grids

$$V_{h,\tilde{h}} = \left\{ \left\{ \sum_{j=1}^{m_{h_e}} \beta_j^e \phi_j^e \right\} \in \prod_{e=1}^{E} V_{h_e} : \{\Lambda^e \boldsymbol{\beta}_e\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}} \right\} \not\subset H^1(\Omega) \tag{41}$$

Conversely, let $\{\boldsymbol{\alpha}_\varepsilon^e\} \in \prod_{e=1}^{E} \mathbf{S}_\varepsilon^{h_e}$, with $\{\Lambda^e \boldsymbol{\alpha}_\varepsilon^e\} \in \mathbf{Q}_{\mathrm{D}_{\tilde{h}}}$, be a solution of primal problem $(\mathrm{V}_{\varepsilon,h,\tilde{h}})$. Then, the residual $\{\boldsymbol{\lambda}_e\} = -\{\mathbf{A}^e \boldsymbol{\alpha}_\varepsilon^e + (1-\rho)\mathbf{H}_\varepsilon^e(\boldsymbol{\alpha}_\varepsilon^e) + \rho \mathbf{H}_\varepsilon^e(\boldsymbol{\alpha}_\varepsilon^e - \boldsymbol{\eta}_\varepsilon^e)\} \in \partial(I_{Q_{\mathrm{D}_{\tilde{h}}}} \circ [\Lambda^e])(\{\boldsymbol{\alpha}_\varepsilon^e\})$ and is necessarily orthogonal to the kernel of the coupling matrices $[\Lambda^e]$ (cf. $(40)_2$); i.e. $\{\boldsymbol{\lambda}_e\} \cdot \{\boldsymbol{\beta}_e\} = 0$, $\forall \{\boldsymbol{\beta}_e\} \in \mathscr{N}([\Lambda^e])$. Hence, since $\mathscr{N}([\Lambda^e]) = \mathscr{R}([\Lambda^{e\mathrm{T}}])^\perp \Leftrightarrow \mathscr{N}([\Lambda^e])^\perp = \mathscr{R}([\Lambda^{e\mathrm{T}}])$, there is a dual vector $\{\boldsymbol{\delta}^e\} \in \prod_{e=1}^{E} \Re^{n_{\tilde{h}_e}}$ such that $\{\boldsymbol{\lambda}_e\} = \{\Lambda^{e\mathrm{T}} \boldsymbol{\delta}^e\}$ and, under the compatibility condition $(\mathrm{C1}_{h,\tilde{h}})$, from relations (39) it follows that $(\{\boldsymbol{\alpha}_\varepsilon^e\}, \{\boldsymbol{\delta}^e\})$ solves macro-hybrid discrete problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$. Therefore, problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$ has a unique solution if, and only if, problem $(\mathrm{V}_{\varepsilon,h,\tilde{h}})$ has a unique solution.

4.2. *Solvability of problem* $(\mathrm{V}_{\varepsilon,h,\tilde{h}})$

Concerning the solvability of primal discrete problem $(\mathrm{V}_{\varepsilon,h,\tilde{h}})$, we first observe that its local subproblems behave as nonlinear advection–diffusion problems whose advective local operators are not all necessarily positive and advection becomes dominant as the penalization parameter $\varepsilon \to 0$. Consequently, local discrete subproblems have to be stabilized for well-posedness and uniform convergence. With respect to the positiveness of the local nonlinear 'advective' terms, through integration by parts we see that they are given by subdomain as well as by internal boundary components: for $\boldsymbol{\beta} \in \mathbf{K}_0^{h_e}$, $e = 1, 2, \ldots, E$,

$$\begin{aligned} &(1-\rho)H_{\varepsilon,i}^e(\boldsymbol{\beta}) + \rho H_{\varepsilon,i}^e(\boldsymbol{\beta} - \boldsymbol{\eta}_\varepsilon^e) \\ &\quad = -\int_{\Omega_e} \left((1-\rho)\partial_y H_\varepsilon \left(\sum_{k=1}^{m_{h_e}} \beta_k \phi_k^e \right) - \rho \partial_y H_\varepsilon \left(\sum_{k=1}^{m_{h_e}} (\beta_k - \eta_{\varepsilon k}^e)\phi_k^e \right) \right) \phi_i^e \, \mathrm{d}\Omega \\ &\qquad + \int_{\Gamma_e} \left((1-\rho)H_\varepsilon \left(\sum_{k=1}^{m_{h_e}} \beta_k \phi_k^e \right) + \rho H_\varepsilon \left(\sum_{k=1}^{m_{h_e}} (\beta_k - \eta_{\varepsilon k}^e) \right) \phi_k^e \right) n_y^e \gamma_{\Gamma_e} \phi_i^e \, \mathrm{d}\Gamma \end{aligned} \tag{42}$$

The subdomain components can be proved all indeed to be positive following Theorem 5.1 in Reference [12]. For the internal boundary components, it is clear that all of them must be equal to zero for positiveness due to their n_y^e-dependence; i.e. the vertical-interface condition

$$\text{(C2)} \quad n_{ef,y} \equiv \nabla y \cdot \mathbf{n}_{ef} = 0, \quad \text{a.e on } \Gamma_{ef}, \; 1 \leqslant e < f \leqslant E$$

must be imposed, restricting nonconforming domain decompositions (19) and (20) to have only vertical interfaces. Hence, assuming condition (C2) is fulfilled, local primal subproblems of macro-hybrid discrete problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$ are well-posed for parallel resolution. Moreover, macro-hybrid primal discrete problem $(\mathrm{V}_{\varepsilon,h,\tilde{h}})$ is then uniquely solvable (cf. Reference [12]).

4.3. *Tabata upwind stabilization*

On the other hand, for uniform convergence, as in Reference [15] we shall apply the Tabata upwind technique [21]. That is, assuming the local triangulations $\mathscr{T}_{h_e}$, $1 \leqslant e \leqslant E$, consist of triangles of the acute type, we approximate nonlinear discrete functions $(35)_2$ through the scheme

$$H^e_{\varepsilon,i}(\boldsymbol{\beta}) \cong \sum_{j=1}^{m_{h_e}} B^e_{ij} H_\varepsilon(\beta_j), \quad \boldsymbol{\beta} \in \Re^{m_{h_e}} \tag{43}$$

with local 'advective' matrices $\mathbf{B}^e$ defined by

$$B^e_{ij} = \begin{cases} \dfrac{1}{3}\mathrm{meas}(S^e_j)\dfrac{\partial \phi^e_i|_{U_j}}{\partial y} & \text{if } U_j \text{ exists} \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

Here $U_j \in \mathscr{T}_{h_e}$ denotes the Tabata upwind finite element associated to the jth vertex of the local triangulation, with respect to the transpose 'flow direction' $\nabla y = (0, 1)$, and S^e_j is the support of function ϕ^e_j.

5. FIXED-POINT RESOLUTION ALGORITHMS

Next, proceeding as for penalized finite element approximations in Reference [15], we first characterize the local penalized discrete problems of macro-hybrid model $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$ as monotone fixed-point problems in accordance with Reference [8]. Also, following the resolvent methodology [16–18], we characterize its synchronization problem as augmented proximation fixed-point problems of two- and three-field type. Then, on the basis of such characterizations, we will be able to introduce iterative resolution algorithms in a sequential form, with penalty-duality schemes for synchronization.

5.1. *Local penalized discrete problems*

For discrete macro-hybrid penalized problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$, let us introduce the local vector functions, for $1 \leqslant i \leqslant m_{h_e}$, $1 \leqslant e \leqslant E$,

$$\begin{aligned} \gamma^e_{\varepsilon,i} &= H_\varepsilon(\alpha^e_{\varepsilon,i}) \\ \lambda^e_{\varepsilon,i} &= H_\varepsilon(\alpha^e_{\varepsilon,i} - \eta^e_{\varepsilon,i}) \end{aligned} \tag{45}$$

that determine numerically as characteristic functions the penalized fresh water flow regions above and below the sea level, respectively. Then, the local penalized discrete problems, with numerical upwinding (43) and (44) take the form,

$$(\tilde{\mathrm{V}}_{\varepsilon,h_e})\quad \begin{cases} \text{Find } \boldsymbol{\alpha}^e_\varepsilon \in \mathbf{S}^{h_e}_\varepsilon \\ \{\mathbf{A}^e\boldsymbol{\alpha}^e_\varepsilon + (1-\rho)\mathbf{B}^e\boldsymbol{\gamma}^e_\varepsilon + \rho\mathbf{B}^e\boldsymbol{\lambda}^e_\varepsilon\}\cdot\boldsymbol{\beta} = -\Lambda^{e\mathrm{T}}\boldsymbol{\delta}^e_\varepsilon\cdot\boldsymbol{\beta} \quad \forall\boldsymbol{\beta}\in\mathbf{K}^{h_e}_0 \end{cases}$$

From the relaxation form of these problems,

$$\begin{aligned} &A^e_{ii}\alpha^e_{\varepsilon,i} + (1-\rho)B^e_{ii}\gamma^e_{\varepsilon,i} + \rho B^e_{ii}\lambda^e_{\varepsilon,i} = c^e_i \\ &c^e_i = -\sum_{j\neq i}^{m_{h_e}}\{A^e_{ij}\alpha^e_{\varepsilon,j} + (1-\rho)B^e_{ij}\gamma^e_{\varepsilon,j} + \rho B^e_{ij}\lambda^e_{\varepsilon,j}\} - \sum_{j=1}^{n_{\tilde{h}_e}}\{\Lambda^e_{ji}\delta^e_{\varepsilon,j}\} \end{aligned} \tag{46}$$

for $i\in\mathscr{I}_{\Omega_e\cup\partial\Omega_{eN}\cup\Gamma_e}$, the index set of the interior and flux boundary degrees of freedom, and under the condition

$$(\mathrm{H3})\quad \varepsilon A^e_{ii} + \rho B^e_{ii} > 0, \qquad i\in\mathscr{I}_{\Omega_e\cup\partial\Omega_{eN}\cup\Gamma_e}$$

the following fixed-point problem characterization can be concluded (see Reference [15]). Here, $\mathscr{I}^+_{\Omega_e\cup\partial\Omega_{e,\mathrm{N}}\cup\Gamma_e}$ and $\mathscr{I}^-_{\Omega_e\cup\partial\Omega_{e,\mathrm{N}}\cup\Gamma_e}$ will denote the index subsets of $\mathscr{I}_{\Omega_e\cup\partial\Omega_{e,\mathrm{N}}\cup\Gamma_e}$ of degrees of freedom above and below the sea level, respectively.

Case 1: For $\eta^e_{\varepsilon,i}+\varepsilon\leqslant\alpha^e_{\varepsilon,i}$, $i\in\mathscr{I}_{\Omega_e\cup\partial\Omega_{e,\mathrm{N}}\cup\Gamma_e}$:

$$\begin{aligned} &\gamma^e_{\varepsilon,i}=1,\quad \lambda^e_{\varepsilon,i}=1,\quad \alpha^e_{\varepsilon,i}=\frac{c^e_i-B^e_{ii}}{A^e_{ii}} \\ &\text{whenever } A^e_{ii}(\eta^e_{\varepsilon,i}+\varepsilon)\leqslant c^e_i-B^e_{ii} \end{aligned} \tag{47}$$

Case 2: For $\varepsilon\leqslant\alpha^e_{\varepsilon,i}<\eta^e_{\varepsilon,i}+\varepsilon$, $i\in\mathscr{I}^+_{\Omega_e\cup\partial\Omega_{eN}\cup\Gamma_e}$:

$$\begin{aligned} &\gamma^e_{\varepsilon,i}=1,\quad \lambda^e_{\varepsilon,i}=\frac{1}{\varepsilon}(\alpha^e_{\varepsilon,i}-\eta^e_{\varepsilon,i}),\quad \alpha^e_{\varepsilon,i}=\frac{c^e_i-B^e_{ii}\{1-\rho\left(1+(1/\varepsilon)\eta^e_{\varepsilon,i}\right)\}}{A^e_{ii}+(1/\varepsilon)\rho B^e_{ii}} \\ &\text{whenever } \varepsilon A^e_{ii}-\frac{1}{\varepsilon}\rho B^e_{ii}\eta^e_{\varepsilon,i}\leqslant c^e_i-B^e_{ii} < A^e_{ii}(\eta^e_{\varepsilon,i}+\varepsilon) \end{aligned} \tag{48}$$

Case 3: For $\eta^e_{\varepsilon,i}\leqslant\alpha^e_{\varepsilon,i}<\varepsilon$, $i\in\mathscr{I}^+_{\Omega_e\cup\partial\Omega_{eN}\cup\Gamma_e}$:

$$\begin{aligned} &\gamma^e_{\varepsilon,i}=\frac{1}{\varepsilon}\alpha^e_{\varepsilon,i},\quad \lambda^e_{\varepsilon,i}=\frac{1}{\varepsilon}(\alpha^e_{\varepsilon,i}-\eta^e_{\varepsilon,i}),\quad \alpha^e_{\varepsilon,i}=\frac{\varepsilon c^e_i+\rho B^e_{ii}\eta^e_{\varepsilon,i}}{\varepsilon A^e_{ii}+B^e_{ii}} \\ &\text{whenever } A^e_{ii}\eta^e_{\varepsilon,i}+B^e_{ii}\left\{\frac{1}{\varepsilon}(1-\rho)\eta^e_{\varepsilon,i}-1\right\}\leqslant c^e_i-B^e_{ii} < \varepsilon A^e_{ii}-\frac{1}{\varepsilon}\rho B^e_{ii}\eta^e_{\varepsilon,i} \end{aligned} \tag{49}$$

Case 4: For $0\leqslant\alpha^e_{\varepsilon,i}<\eta^e_{\varepsilon,i}$, $i\in\mathscr{I}^+_{\Omega_e\cup\partial\Omega_{eN}\cup\Gamma_e}$:

$$\begin{aligned} &\gamma^e_{\varepsilon,i}=\frac{1}{\varepsilon}\alpha^e_{\varepsilon,i},\quad \lambda^e_{\varepsilon,i}=0,\quad \alpha^e_{\varepsilon,i}=\frac{\varepsilon c^e_i}{\varepsilon A^e_{ii}+(1-\rho)B^e_{ii}} \\ &\text{whenever } -B^e_{ii}\leqslant c^e_i-B^e_{ii}<A^e_{ii}\eta^e_{\varepsilon,i}+B^e_{ii}\left\{\frac{1}{\varepsilon}(1-\rho)\eta^e_{\varepsilon,i}-1\right\} \end{aligned} \tag{50}$$

Case 5: For $\eta^e_{\varepsilon,i} \leqslant \alpha^e_{\varepsilon,i} < \eta^e_{\varepsilon,i} + \varepsilon$, $i \in \mathscr{I}^-_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$:

$$\begin{aligned}&\gamma^e_{\varepsilon,i} = 1, \quad \lambda^e_{\varepsilon,i} = \frac{1}{\varepsilon}(\alpha^e_{\varepsilon,i} - \eta^e_{\varepsilon,i}), \quad \alpha^e_{\varepsilon,i} = \frac{c^e_i - B^e_{ii}\left\{1 - \rho\left(1 + (1/\varepsilon)\eta^e_{\varepsilon,i}\right)\right\}}{A^e_{ii} + (1/\varepsilon)\rho B^e_{ii}} \\ &\text{whenever } A^e_{ii}\eta^e_{\varepsilon,i} - \rho B^e_{ii} \leqslant c^e_i - B^e_{ii} < A^e_{ii}(\eta^e_{\varepsilon,i} + \varepsilon)\end{aligned} \tag{51}$$

Case 6: For $\varepsilon \leqslant \alpha^e_{\varepsilon,i} < \eta^e_{\varepsilon,i}$, $i \in \mathscr{I}^-_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$:

$$\begin{aligned}&\gamma^e_{\varepsilon,i} = 1, \quad \lambda^e_{\varepsilon,i} = 0, \quad \alpha^e_{\varepsilon,i} = \frac{c^e_i - (1-\rho)B^e_{ii}}{A^e_{ii}} \\ &\text{whenever } \varepsilon A^e_{ii} - \rho B^e_{ii} \leqslant c^e_i - B^e_{ii} < A^e_{ii}\eta^e_{\varepsilon,i} - \rho B^e_{ii}\end{aligned} \tag{52}$$

Case 7: For $0 \leqslant \alpha^e_{\varepsilon,i} < \varepsilon$, $i \in \mathscr{I}^-_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$:

$$\begin{aligned}&\gamma^e_{\varepsilon,i} = \frac{1}{\varepsilon}\alpha^e_{\varepsilon,i}, \quad \lambda^e_{\varepsilon,i} = 0, \quad \alpha^e_{\varepsilon,i} = \frac{\varepsilon c^e_i}{\varepsilon A^e_{ii} + (1-\rho)B^e_{ii}} \\ &\text{whenever } -B^e_{ii} \leqslant c^e_i - B^e_{ii} < \varepsilon A^e_{ii} - \rho B^e_{ii}\end{aligned} \tag{53}$$

Case 8: For $\alpha^e_{\varepsilon,i} < 0$, $i \in \mathscr{I}_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$:

$$\begin{aligned}&\gamma^e_{\varepsilon,i} = 0, \quad \lambda^e_{\varepsilon,i} = 0, \quad \alpha^e_{\varepsilon,i} = \frac{c^e_i}{A^e_{ii}} \\ &\text{whenever } c^e_i - B^e_{ii} < -B^e_{ii}\end{aligned} \tag{54}$$

In summary, local penalized discrete problems $(\tilde{\mathrm{V}}_{\varepsilon,h_e})$, $1 \leqslant e \leqslant E$, are characterized by the local fixed-point problems

$$(\alpha^e_{\varepsilon,i}, \gamma^e_{\varepsilon,i}(\alpha^e_{\varepsilon,i}), \lambda^e_{\varepsilon,i}(\alpha^e_{\varepsilon,i})) = F^e_{\varepsilon,i}(\{\alpha^e_{\varepsilon,j}, \gamma^e_{\varepsilon,j}(\alpha^e_{\varepsilon,j}), \lambda^e_{\varepsilon,j}(\alpha^e_{\varepsilon,j})\}^{m_h}_{j\neq i}) \tag{55}$$

relative to the interior and boundary flux degrees of freedom $i \in \mathscr{I}_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$, by formulae (47)–(54). Then, as a natural iterative resolution algorithm for the local filtration problems, we have the following one with relaxation: for $l \geqslant 0$ and $i \in \mathscr{I}_{\Omega_e \cup \partial\Omega_{eN} \cup \Gamma_e}$,

$$(\alpha^e_{\varepsilon,i,l+1}, \gamma^e_{\varepsilon,i,l+1}(\alpha^e_{\varepsilon,i,l+1}), \lambda^e_{\varepsilon,i,l+1}(\alpha^e_{\varepsilon,i,l+1})) = F^e_{\varepsilon,i}\begin{pmatrix}\{\alpha^e_{\varepsilon,j,l+1}, \gamma^e_{\varepsilon,j,l+1}(\alpha^e_{\varepsilon,j,l+1}), \lambda^e_{\varepsilon,j,l+1}(\alpha^e_{\varepsilon,j,l+1})\}^{i-1}_{j=1} \\ \{\alpha^e_{\varepsilon,j,l}, \gamma^e_{\varepsilon,j,l}(\alpha^e_{\varepsilon,j,l}), \lambda^e_{\varepsilon,j,l}(\alpha^e_{\varepsilon,j,l})\}^{m_h}_{j=i+1}\end{pmatrix} \tag{56}$$

with given initial vector $\{\boldsymbol{\alpha}^e_{\varepsilon,0}\} \in \prod^E_{e=1}\mathbf{S}^{h_e}_\varepsilon$; we refer to Reference [10] for other alternative algorithms.

5.2. *Interface synchronization problem*

We next turn to the synchronization process of the discrete macro-hybrid penalized variational problem $(\mathrm{MH}_{\varepsilon,h,\tilde{h}})$. For its proximation fixed-point characterization, we first rewrite it in its dual subdifferential form (cf. (39))

$$(\mathrm{DS}_{\varepsilon,\tilde{h}}) \quad \begin{cases}\text{Find } \{\boldsymbol{\delta}^e_\varepsilon\} \in \prod\limits^E_{e=1}\Re^{n_{\tilde{h}_e}} \\ \{\Lambda^e\boldsymbol{\alpha}^e_\varepsilon\} \in \boldsymbol{\partial} I_{Q^*_{\mathrm{N}_{\tilde{h}}}}(\{\boldsymbol{\delta}^e_\varepsilon\})\end{cases}$$

where $I_{Q^*_{\mathrm{N}_{\tilde{h}}}}$ is the indicator functional of the coordinate finite element dual transmission admissibility subspace $\mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$ of (36). Now, we can identified such a synchronization process with a fixed-point problem in terms of the resolvent of the dual subdifferential $\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}}$,

$$\mathbf{J}^r_{\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}}} \equiv (\mathbf{I} + r\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}})^{-1} \tag{57}$$

which is a single valued firm contraction [28]. Here, $\mathbf{I}$ is the identity block matrix of order $n_{\tilde{h}} = \sum_{e=1}^{E} n_{\tilde{h}_e}$, and $r>0$ is a positive fixed parameter. Indeed, expressing the subdifferential equation of $(\mathrm{DS}_{\varepsilon,\tilde{h}})$ in the augmented equivalent form $\{\boldsymbol{\delta}^e_\varepsilon\} + r\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\} \in (\mathbf{I} + r\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}})(\{\boldsymbol{\delta}^e_\varepsilon\})$, we have its characterization

$$\{\boldsymbol{\delta}^e_\varepsilon\} = \mathbf{J}^r_{\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}}}(\{\boldsymbol{\delta}^e_\varepsilon\} + r\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\}) \tag{58}$$

Further, resolvent operator (57) has the primal projection interpretation [29],

$$\begin{aligned}\mathbf{J}^r_{\partial I_{Q^*_{\mathrm{N}_{\tilde{h}}}}}(\{\mathbf{v}^e_\varepsilon\}) &= (\mathbf{I} - \mathrm{Proj}_{Q_{\mathrm{D}_{\tilde{h}}}})(\{\mathbf{v}^e_\varepsilon\}) \\ &= \{\mathbf{v}^e_\varepsilon\} - \arg\left(\inf_{\{\mu^e\}\in Q_{\mathrm{D}_{\tilde{h}}}} \frac{1}{2}\|\{\boldsymbol{\mu}^e\} - \{\mathbf{v}^e_\varepsilon\}\|^2_{\Re^{n_{\tilde{h}}}}\right)\end{aligned} \tag{59}$$

where $\mathbf{Q}_{\mathrm{D}_{\tilde{h}}}$ is the coordinate finite element primal transmission admissibility subspace of (37), orthogonal to $\mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$.

Therefore, on the basis of fixed-point problem characterization (58), proximal-point approximations of synchronization problem $(\mathrm{DS}_{\varepsilon,\tilde{h}})$ can be derived. In particular, utilizing projection interpretation (59), we incorporate to our resolution procedure the following iterative algorithm: for $m \geqslant 0$,

$$\{\boldsymbol{\delta}^e_{\varepsilon,m+1}\} = (\mathbf{I} - \mathrm{Proj}_{Q_{\mathrm{D}_{\tilde{h}}}})(\{\boldsymbol{\delta}^e_{\varepsilon,m}\} + r\{\Lambda^e \boldsymbol{\alpha}^e_{\varepsilon,m+1}\}) \tag{60}$$

with initial vectors $(\{\boldsymbol{\alpha}^e_{\varepsilon,1}\}, \{\boldsymbol{\delta}^e_{\varepsilon,0}\}) \in \prod_{e=1}^{E} \mathbf{S}^{h_e}_\varepsilon \times \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$. Algorithm (60) corresponds to the dual part of the penalty-duality algorithms *ALG4* and *ALG5*, introduced in Reference [30] for unilateral contact problems, and studied for monotone variational inequalities in References [16–18, 29].

On the other hand, we can reformulate equivalently dual synchronizing problem $(\mathrm{DS}_{\varepsilon,\tilde{h}})$ through its primal version (cf. (39)), by introducing the intermediate primal vector $\{\boldsymbol{\vartheta}^e_\varepsilon\} = \{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\}$, related to the internal boundary pressures, as follows:

$$(\mathrm{PS}_{\varepsilon,\tilde{h}}) \quad \begin{cases} \text{Find } \{\boldsymbol{\vartheta}^e_\varepsilon\} \in \prod_{e=1}^{E} \Re^{n_{\tilde{h}_e}} \text{ and } \{\boldsymbol{\delta}^e_\varepsilon\} \in \prod_{e=1}^{E} \Re^{n_{\tilde{h}_e}} \\ \{\boldsymbol{\delta}^e_\varepsilon\} \in \partial I_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\boldsymbol{\vartheta}^e_\varepsilon\}) \\ \{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\} - \{\boldsymbol{\vartheta}^e_\varepsilon\} = \{\mathbf{0}^e\} \end{cases}$$

Then, expressing the primal subdifferential equation in the augmented form $\{\boldsymbol{\delta}^e_\varepsilon\} + r/2\{\Lambda^e \boldsymbol{\alpha}^e_\varepsilon\} \in (\partial I_{Q_{\mathrm{D}_{\tilde{h}}}} + r/2\mathbf{I})(\{\boldsymbol{\vartheta}^e_\varepsilon\})$, we introduce the alternative resolution iterative algorithm for synchronization: for $m \geqslant 0$,

$$
\begin{aligned}
&\{\boldsymbol{\delta}^e_{\varepsilon,m} + r/2\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m}\} \in (\partial I_{Q_{\mathrm{D}_{\tilde{h}}}} + r/2\mathbf{I})(\{\boldsymbol{\vartheta}^e_{\varepsilon,m+1}\}) \\
&\boldsymbol{\delta}^e_{\varepsilon,m+1/2} = \boldsymbol{\delta}^e_{\varepsilon,m} + r/2(\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m} - \boldsymbol{\vartheta}^e_{\varepsilon,m+1}), \quad e=1,\ldots,E \\
&\boldsymbol{\delta}^e_{\varepsilon,m+1} = \boldsymbol{\delta}^e_{\varepsilon,m+1/2} + r/2(\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m+1} - \boldsymbol{\vartheta}^e_{\varepsilon,m+1}), \quad e=1,\ldots,E
\end{aligned}
\tag{61}
$$

with initial vectors $(\{\boldsymbol{\alpha}^e_{\varepsilon,0}\},\{\boldsymbol{\delta}^e_{\varepsilon,0}\}) \in \prod_{e=1}^{E}\mathbf{S}^{h_e}_{\varepsilon} \times \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$. This algorithm corresponds to the dual macro-hybrid part of the operator splitting algorithm *ALG3* presented in Reference [17] for macro-hybrid mixed variational inequalities.

5.3. Parallel relaxation penalty-duality algorithms

We are now in a position to associate to macro-hybrid penalized discrete problem $(\tilde{\mathrm{MH}}_{\varepsilon,h,\tilde{h}}) = (\{\tilde{\mathrm{V}}_{\varepsilon,h_e}\},\mathrm{DS}_{\varepsilon,\tilde{h}})$ resolution recursive processes of a relaxation penalty-duality type in parallel. First, combining scheme (56) for the nonlinear local penalized discrete problems $(\{\tilde{\mathrm{V}}_{\varepsilon,h_e}\})$ and algorithm (60) for the dual synchronization $(\mathrm{DS}_{\varepsilon,\tilde{h}})$, we have the following:

Algorithm 1
Given $\{\boldsymbol{\alpha}^e_{\varepsilon,0}\} \in \prod_{e=1}^{E}\mathbf{S}^{h_e}_{\varepsilon}$ and $\{\boldsymbol{\delta}^e_{\varepsilon,0}\} \in \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$,
known $\{\boldsymbol{\alpha}^e_{\varepsilon,m}\}$ and $\{\boldsymbol{\delta}^e_{\varepsilon,m}\}$, $m \geqslant 0$,
calculate in parallel $\boldsymbol{\alpha}^e_{\varepsilon,m+1} \in \mathbf{S}^{h_e}_{\varepsilon}$, through the local relaxation scheme (56),
$e=1,2,\ldots,E$:
$(\tilde{\mathbf{A}}^e_r\boldsymbol{\alpha}^e_{\varepsilon,m+1} + (1-\rho)\mathbf{B}^e\boldsymbol{\gamma}^e_{\varepsilon,m+1} + \rho\mathbf{B}^e\boldsymbol{\lambda}^e_{\varepsilon,m+1})\cdot\boldsymbol{\beta} = -\Lambda^{e\mathrm{T}}\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m}\cdot\boldsymbol{\beta},\ \forall\boldsymbol{\beta}\in\mathbf{K}^{h_e}_0$,
with $\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m}$ defined by
$\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m} = \boldsymbol{\delta}^e_{\varepsilon,m} - \mathrm{Proj}_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\boldsymbol{\delta}^f_{\varepsilon,m} + r\Lambda^f\boldsymbol{\alpha}^f_{\varepsilon,m}\})_e$;
and $\{\boldsymbol{\delta}^e_{\varepsilon,m+1}\}$ according to the synchronization
$\{\boldsymbol{\delta}^e_{\varepsilon,m+1}\} = \{\boldsymbol{\delta}^e_{\varepsilon,m} + r\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m+1}\} - \mathrm{Proj}_{Q_{\mathrm{D}_{\tilde{h}}}}(\{\boldsymbol{\delta}^e_{\varepsilon,m} + r\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m+1}\})$

Here, $\tilde{\mathbf{A}}^e_r$ denotes the augmented version of matrix $\mathbf{A}^e$,

$$
\tilde{\mathbf{A}}^e_r = \mathbf{A}^e + r\Lambda^{e\mathrm{T}}\Lambda^e, \quad e=1,2,\ldots,E \tag{62}
$$

From the convergence analysis of this algorithm, as algorithm *ALG5* in Reference [17, Remark 5.5], we shall just mention that under maximal monotonicity conditions convergence is guaranteed for any fixed $r>0$. Experimentally, parameter r can be chosen in order to speed up the convergence.

On the other hand, for the three-field version $(\widetilde{\mathrm{MH}}_{\varepsilon,h,\tilde{h}}) = (\{\tilde{\mathrm{V}}_{\varepsilon,h_e}\},\mathrm{PS}_{\varepsilon,\tilde{h}})$ of the macro-hybrid penalized discrete problem, combining relaxation scheme (56) and synchronizing penalty-duality algorithm (61) we obtain an alternative algorithm.

Algorithm 2
Given $\{\boldsymbol{\alpha}^e_{\varepsilon,0}\} \in \prod_{e=1}^{E}\mathbf{S}^{h_e}_{\varepsilon}$ and $\{\boldsymbol{\delta}^e_{\varepsilon,0}\} \in \mathbf{Q}^*_{\mathrm{N}_{\tilde{h}}}$,
known $\{\boldsymbol{\alpha}^e_{\varepsilon,m}\}$ and $\{\boldsymbol{\delta}^e_{\varepsilon,m}\}$, $m \geqslant 0$,

calculate $\{\boldsymbol{\vartheta}^e_{\varepsilon,m+1}\}$ according to the synchronization
$\{\boldsymbol{\vartheta}^e_{\varepsilon,m+1}\} = \mathrm{Proj}_{Q_{D_{\tilde{h}}}}(\{2/r\boldsymbol{\delta}^e_{\varepsilon,m} + \Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m}\});$
and, in parallel, $\boldsymbol{\delta}^e_{\varepsilon,m+1/2}$, $\boldsymbol{\alpha}^e_{\varepsilon,m+1} \in \mathbf{S}^{h_e}_{\varepsilon}$ through the local relaxation scheme (56), and $\boldsymbol{\delta}^e_{\varepsilon,m+1}$, $e = 1, 2, \ldots, E$:

$$\boldsymbol{\delta}^e_{\varepsilon,m+1/2} = \boldsymbol{\delta}^e_{\varepsilon,m} + r/2(\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m} - \boldsymbol{\vartheta}^e_{\varepsilon,m+1}),$$

$$(\tilde{\mathbf{A}}^e_{r/2}\boldsymbol{\alpha}^e_{\varepsilon,m+1} + (1-\rho)\mathbf{B}^e\boldsymbol{\gamma}^e_{\varepsilon,m+1} + \rho\mathbf{B}^e\Lambda^e_{\varepsilon,m+1})\cdot\boldsymbol{\beta} = -\Lambda^{e\mathrm{T}}\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m+1/2}\cdot\boldsymbol{\beta},\ \forall\boldsymbol{\beta}\in\mathbf{K}^{h_e}_0,$$

with $\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m+1/2}$ defined by

$$\tilde{\boldsymbol{\delta}}^e_{\varepsilon,m+1/2} = \boldsymbol{\delta}^e_{\varepsilon,m+1/2} - r/2\boldsymbol{\vartheta}^e_{\varepsilon,m+1},$$

$$\boldsymbol{\delta}^e_{\varepsilon,m+1} = \boldsymbol{\delta}^e_{\varepsilon,m+1/2} + r/2(\Lambda^e\boldsymbol{\alpha}^e_{\varepsilon,m+1} - \boldsymbol{\vartheta}^e_{\varepsilon,m+1})$$

Here, $\tilde{\mathbf{A}}^e_{r/2}$ denotes the augmented version of matrix $\mathbf{A}^e$, (62), but with parameter r replaced by $r/2$. This algorithm corresponds to algorithm *ALG3* studied in Reference [17], which under maximal monotonicity conditions converges for any fixed $r > 0$ (see Reference [17, Remark 5.10]). Also, experimentally, parameter r can be chosen to speed up convergence.

6. NUMERICAL EXPERIMENTS

Testing the macro-hybrid penalized finite element discrete model ($\mathrm{MH}_{\varepsilon,h,\tilde{h}}$) and the relaxation penalty-duality schemes Algorithms 1 and 2, we have considered coastal aquifer sections with general geometries subject to conditions (H1) and (H2), decomposed into connected disjoint subdomains satisfying the local conditions (C2) of vertical interfaces. Discretizations are given in terms of subdomain grids, not matching at the interfaces, and independent interface grids no finer than the traces of the neighbouring subdomain grids for avoiding locking phenomenon (see Figures 3 and 4 for the cases of one and three reservoirs, respectively). The density parameter (5), $\rho = -0.025$. The boundary conditions below the sea level are impervious at

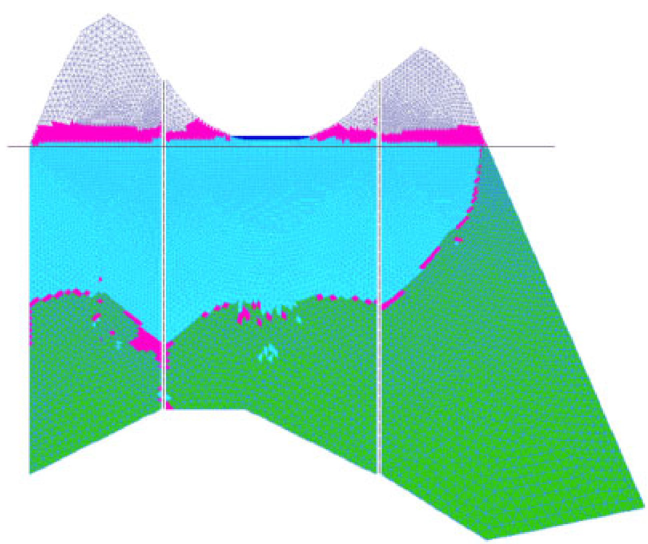

Figure 3. Case 1: numerical flow region in a coastal aquifer section with one reservoir.

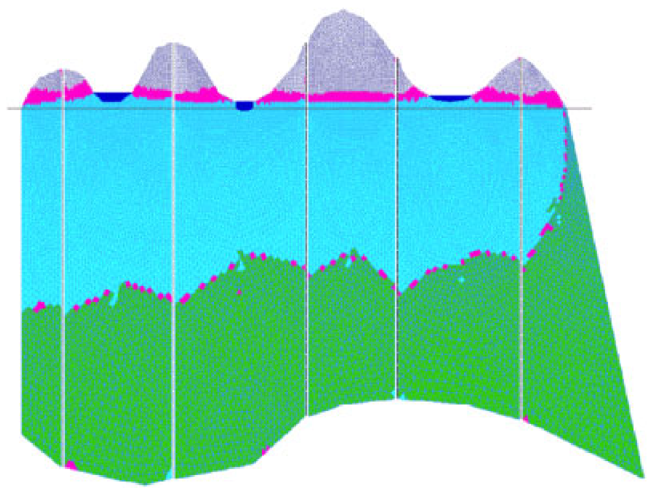

Figure 4. Case 2: numerical flow region in a coastal aquifer section with three reservoirs.

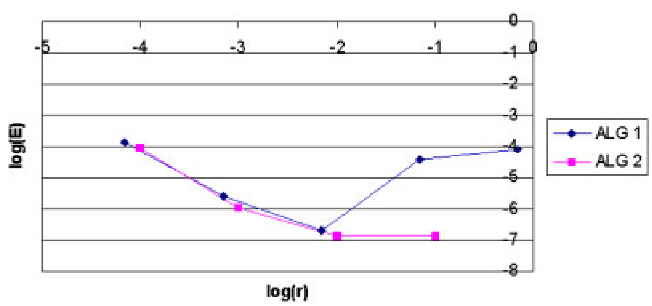

Figure 5. A relation between proximation parameter r and achieved pressure precision.

the bottom and at the left side, and hydrostatic under seawater at the right side; above the sea level they are open to the air, with corresponding seepage positive constraint of outflow, and hydrostatic at the bottom of the fresh water reservoirs with specified hydraulic charges $\hat{h}$.

As nonconforming multidomain finite element approximations compatible in the sense of inf–sup condition (38) [23], we have considered local conforming subdomain pressure spaces V_{h_e}, (31), of a piecewise linear type, and internal boundary flux spaces $B_{\tilde{h}_e}$, (32), of a piecewise constant type, the latter coinciding along the vertical interfaces in agreement with definition (36). In the selection of the penalization parameter $\varepsilon > 0$, we have numerically determined smallest values that satisfy the relaxation condition (H3) in a global and in a local sense, being of an $O(10^{-2})$ and observing non important improvement in utilizing local instead of global values. On the other hand, for Algorithms 1 and 2, we have experimentally studied the influence of the proximation parameter r, showing in Figure 5 its relation with achieved precisions on the pressure field through 50 synchronizing iterations, and in Figure 6 with the number of synchronizing iterations up to an error tolerance of 10^{-6}. From these two graphics

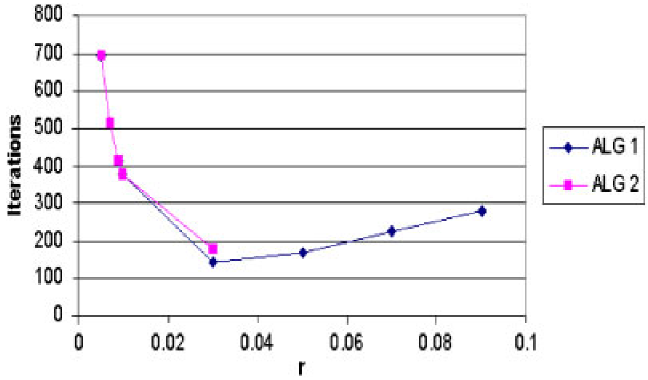

Figure 6. A relation between proximation parameter r and number of synchronizing iterations.

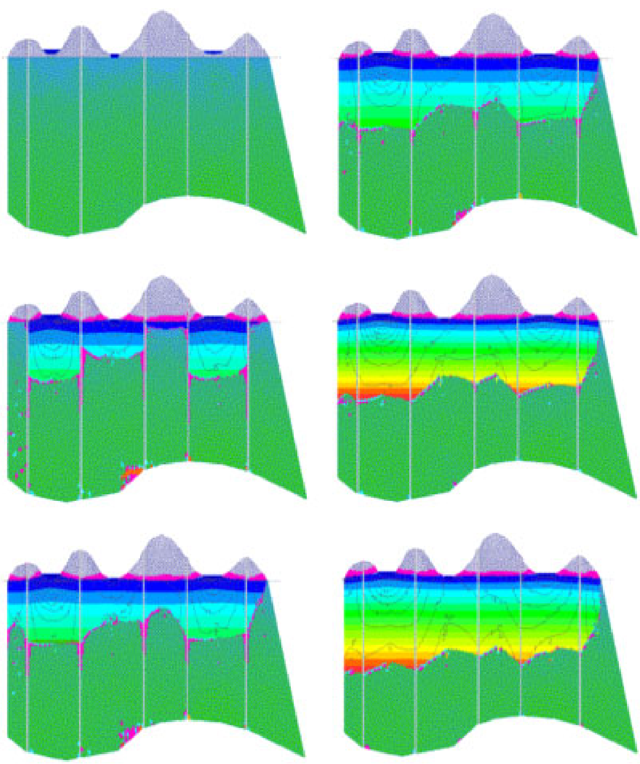

Figure 7. Synchronizing iterative process with level curves of hydraulic charge and pressure distribution.

'optimal' r-parameter values can be determined, resulting Algorithm 2 the most precise while Algorithm 1 the fastest. Figure 7 depicts a synchronizing iterative process of the three-reservoir case, with level curves of the hydraulic charge and pressure distribution, where an experimental 'optimal' r-proximation parameter has been used in accordance with Figure 6.

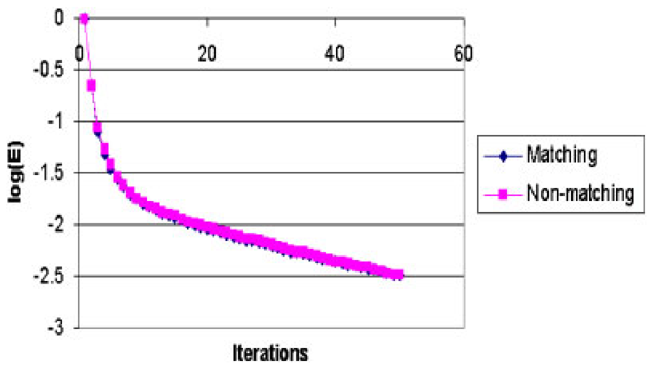

Figure 8. Asymptotic behaviour of the internal boundary pressure error.

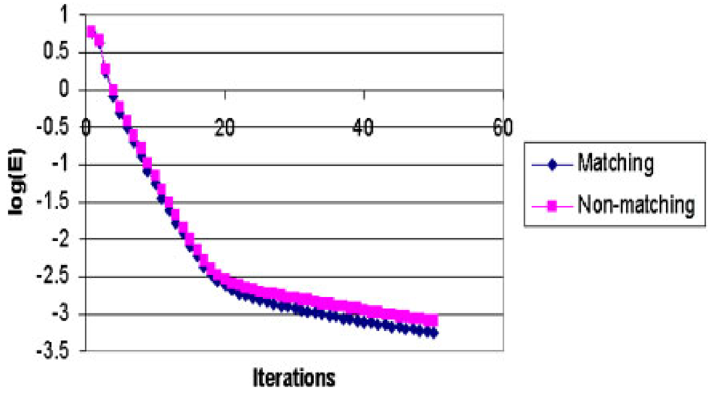

Figure 9. Asymptotic behaviour of the interface synchronizing flux error.

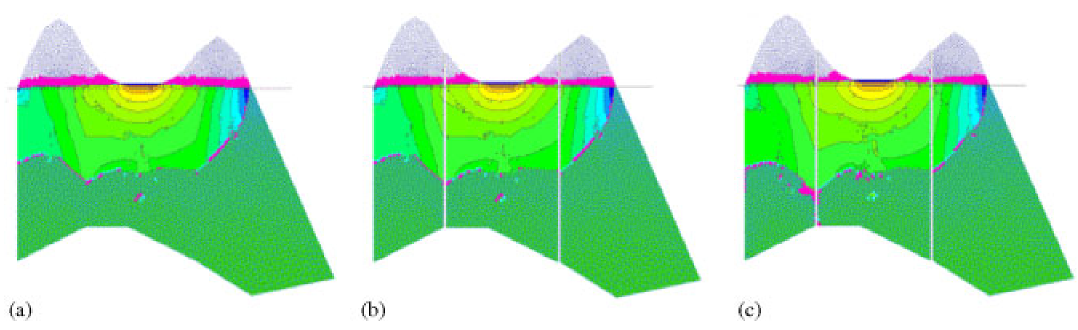

Figure 10. Penalized numerical simulations: (a) conforming primal solution; (b) macro-hybrid solution with matching grids; and (c) macro-hybrid solution with non-matching grids.

With respect to the error behaviour in the relaxation penalty-duality iterative resolution processes, we present in Figure 8 the evolution of internal boundary pressure error at a vertical interface and in Figure 9 of interface synchronizing flux, exhibiting the relative precision of macro-hybrid discretizations with matching and non-matching interface grids.

Finally, in Figure 10 we show the good agreement of a macro-hybrid penalized numerical simulation for the one-reservoir case, with matching and non-matching grids, in contrast with the corresponding globally conforming primal penalized numerical solution without domain decomposition according to Reference [15].

7. CONCLUSIONS

Macro-hybrid penalized variational models of steady Darcian filtration with seawater intrusion have been studied, in accordance with Brezis–Kinderlehrer–Stampacchia variational approach. Subdifferential nonoverlapping subdomain synchronization has been the basis for macro-hybrid formulation and algorithmic development, innovating the standard variational bilinear procedure of macro-hybridization. Composition duality principles have been established for the solvability analysis, with the surjectivity of the internal boundary trace operator as the compatibility condition, operator version of the classical inf–sup hybrid condition. Nonoverlapping domain decompositions have to be restricted to have only vertical interfaces, in order to guarantee the positiveness of the advection terms at a subdomain level. Lastly, this variational theory has been shown to produce natural globally nonconforming discretizations as well as effective parallel proximal-point algorithms.

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